THE YONEDA LEMMA AND (CO)LIMITS
STUDENT SEMINAR ON ∞-CATEGORIES, WINTER 2024/25
6TH DECEMBER 2024

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ABSTRACT. These are the author's notes for a talk in the student seminar on ∞-categories organized by Jonas Heintze in the 2024/25 Winter Semester on the Yoneda lemma and (co)limits.

PRELIMINARIES

We largely follow Wagner's notes [Wag24] in Sections 1 to 4, refering additionally to [You] in Section 1. Section 5 provides an alternative (perhaps more conventional) definition of limits and colimits.

Acknowledgements. I would be remiss not to express my thanks to Jonas for giving me the opportunity to speak in his seminar. I also thank Sam Brinkerhoff for his feedback on a preliminary version of Section 1.

I first learned much of this material from a course given by Prof. M. Hopkins while visiting Harvard in the Fall of 2023, and have benefited greatly from conversations about things ∞-categorical with numerous individuals – in particular the fellow participants at the 2024 Park City Mathematics Institute and my classmates here in Bonn.

1. AN APPLICATION OF STRAIGHTENING-UNSTRAIGTENING: STACKS

The content of this section is an expanded and intiuitionistic variant of [Wag24, §5.7] using [You]. Everything in this section should be taken with heavy skepticism.

We consider an application of the straightening-unstraightening correspondence to the theory of algebraic stacks, a generalization of schemes. The reader unfamiliar with algebraic stacks should be able to replace "scheme" with a sufficiently reasonable category of geometric spaces, or skip immediately to Section 2.

One place where stacks arise is in the consideration of moduli problems – ie. the construction of parameter spaces for schemes of a certain sort. Let X be a scheme and G a group scheme acting on X. One is often motivated to study the "quotient" $[X/G]$, but this may not exist in the category of schemes.

Note that any discrete group can be made into a group scheme by taking finite disjoint unions of $\mathrm{Spec}(\mathbb{Z})$.

Grothendieck's functor of points perspective tells us that we should instead study the collection of maps to the scheme of interest – that is, functors $\mathsf{Sch}^{\mathsf{Opp}} \to \mathsf{Sets}$. We can expand this to the study of functors $\mathsf{Sch}^{\mathsf{Opp}} \to \mathsf{Grpd}$, taking sets as groupoids with the only morphisms given by identities. Whatever $[X/G]$ may be, it admits a morphism from X with fibers given by G-orbits and any $Y \to X$ descends to a map

$[X/G]$ if G acts trivially on Y and the map $Y \to X$ is G-equivariant. We would thus expect a close connection between the following:

- $[X/G]$.
- (Nice) schemes Y with a map to X "sufficiently compatible" with the G-action.
- (Pre)sheaves on Sch_X "sufficiently compatible" with the G-action.

Taking the third perspective, the desiderata above is summarized in the construction of G-torsors, which we omit. As it turns out, for each X-scheme Y, there is a groupoid of G-torsors over Y that assembles into a right fibration $\mathsf{E} \to \mathsf{Sch}_{/X}$ – the fiber over each X-scheme Y is the groupoid of G-torsors over Y. The straightening of this right fibration is a functor $\mathsf{Sch}_{/X}^{\mathsf{Opp}} \to \mathsf{Grpd}$ which produces the "correct" representing functor for $[X/G]$ whose Y-valued points are G-bundles on Y with an equivariant map to X.

More precisely, after passing to nerves and taking Grpd as a $(2,1)$-truncated $(\infty,1)$-category.

2. The Yoneda Lemma

This section, modulo opposites, follows [Wag24, §5.3] fairly closely.

The 1-categorical Yoneda lemma tells us that for a category C and $A \in \mathsf{C}$ that $\mathrm{Mor}_{\mathrm{Fun}(\mathsf{C}^{\mathsf{Opp}},\mathsf{Sets})}(h_A, F) \leftrightarrow F(A)$ as sets. Running everything through the ∞-dictionary, we would expect that $\mathrm{Mor}_{\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}},\mathsf{Ani})}(h_x, F) \simeq F(x)$. This is in fact true, but the proof is a fair amount more involved than in the 1-categorical case since composition being "on the nose" forces certain choices that one can check gives the right answer. Unfortunately the fact that compositions are only defined up to homotopy in the quasicategorical setting means that the proof does not generalize. The upshot is that the straightening-unstraightening correspondence gives the right amount of rigidity for things to work out.

To show the Yoneda lemma, we will use the following result as a black box.

Lemma 2.1 ([Wag24, Thm. 5.19, Lem. 5.20])**.** *Let $\mathcal{C}$ be a quasicategory, $x \in \mathcal{C}$ and $\mathcal{E} \to \mathcal{C}$ any right fibration over $\mathcal{C}$. The diagram*

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}_{/x}, \mathcal{E}) & \longrightarrow & \mathrm{Fun}(\mathcal{C}_{/x}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{C} \end{array} \tag{2.1}$$

is homotopy Cartesian and restricts to the homotopy Cartesian diagram

$$\begin{array}{ccc} \mathrm{coreFun}(\mathcal{C}_{/x}, \mathcal{E}) & \longrightarrow & \mathrm{coreFun}(\mathcal{C}_{/x}, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{core}(\mathcal{E}) & \longrightarrow & \mathrm{core}(\mathcal{C}) \end{array} \tag{2.2}$$

on cores.

As always, $h_x = \mathrm{Mor}_{\mathcal{C}}(-, x)$.

With everything we have set up so far, we can prove the Yoneda lemma.

This here is the opposite of [Wag24, Thm. 5.19].

Theorem 2.2 (Yoneda; [Wag24, Thm. 5.19])**.** Let $\mathcal{C}$ be a quasicategory, $x \in \mathcal{C}$, and $F : \mathcal{C}^{\mathsf{Opp}} \to \mathsf{Ani}$ a functor. There is an equivalence of animae

$$\mathrm{Mor}_{\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}},\mathsf{Ani})}(h_x, F) \longrightarrow F(x).$$

Proof Outline. Let F be as above and $p : \mathcal{E} \to \mathcal{C}$ its (right/Cartesian) unstraightening. Under this construction, the representable functor h_x unstraightens to the source functor $s : \mathcal{C}_{/x} \to \mathcal{C}$. We get equivalences of animae

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}},F)}(h_x, F) &\simeq \mathrm{Mor}_{\mathsf{RFib}(\mathcal{C})}([s : \mathcal{C}_{/x} \to \mathcal{C}], [p : \mathcal{E} \to \mathcal{C}]) \\ &\simeq \mathrm{Mor}_{\mathsf{Cat}_{\infty/\mathcal{C}}}([s : \mathcal{C}_{/x} \to \mathcal{C}], [p : \mathcal{E} \to \mathcal{C}]) \end{aligned}$$

where the inclusion $\mathsf{RFib}(\mathcal{C}) \to \mathsf{Cat}_{\infty/\mathcal{C}}$ is fully faithful.

That $\mathsf{RFib}(\mathcal{C}) \to \mathsf{Cat}_{\infty/\mathcal{C}}$ is fully faithful is morally expected as being a right fibration is a property of quasicategories over $\mathcal{C}$.

By [Wag24, Corr. 5.15], the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathsf{Cat}_{\infty/\mathcal{C}}}([s : \mathcal{C}_{/x} \to \mathcal{C}], [p : \mathcal{E} \to \mathcal{C}]) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{C}_{/x}, \mathcal{E}) & \longrightarrow & \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{C}_{/x}, \mathcal{C}) \end{array} \tag{2.3}$$

The point inclusion on the right picks out s.

is homotopy Cartesian. By pasting with the square (2.2), we extend this to

$$\begin{array}{ccc} \mathrm{Mor}_{\mathsf{Cat}_{\infty/\mathcal{C}}}([s : \mathcal{C}_{/x} \to \mathcal{C}], [p : \mathcal{E} \to \mathcal{C}]) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{core}(\mathcal{E}) & \longrightarrow & \mathrm{core}(\mathcal{C}). \end{array}$$

Right fibrations are in particular isofibrations (vis. [Ker24, Tag 01GP]) and thus $\mathrm{core}(\mathcal{E}) \to \mathrm{core}(\mathcal{C})$ is a Kan fibration so by model category theory [Wag24, Cons. 5.12] – which states that pullbacks along diagrams with one leg a Kan fibration agrees with the pullback in quasicategories – implies that $\mathrm{Mor}_{\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}},F)}(h_x, F)$ is given by the fibered product $\mathrm{core}(\mathcal{E}) \times_{\mathrm{core}(\mathcal{C})} \{x\}$. We can then compute

Recall here $\mathrm{Mor}_{\mathsf{Cat}_\infty}(-,-)$ is computed as $\mathrm{coreFun}(-,-)$ and the right arrow picks out $x \in \mathrm{core}(\mathcal{C})$.

$$\begin{aligned} \mathrm{core}(\mathcal{E}) \times_{\mathrm{core}(\mathcal{C})} \{x\} &\cong \mathrm{core}(\mathcal{E} \times_{\mathcal{C}} \{x\}) && \text{(2.1) restricts to (2.2)} \\ &\cong \mathcal{E} \times_{\mathcal{C}} \{x\} && \text{pullback over (2.3) pasted with (2.1)} \\ &\simeq F(x) \end{aligned}$$

as desired. ■

Moreover, this construction gives a fully faithful embedding of $\mathcal{C}$ into $\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}}, \mathsf{Ani})$.

Proposition 2.3 ([Wag24, Corr. 5.27])**.** The Yoneda embedding $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}}, \mathsf{Ani})$ by $x \mapsto \mathrm{Mor}_{\mathcal{C}}(-, x)$ is fully faithful.

Proof. Full faithfullness is a formal consequence of Theorem 2.2, but that this is induced by the Yoneda embedding is not *prima facie* evident. Doing so requires the twisted arrow construction of [Wag24, §5.21], which we omit. ■

3. Adjoint Functors

This section follows [Wag24, §6.1] fairly closely.

Definition 3.1 (Right Adjoint Object; [Wag24, Def. 6.1])**.** Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of quasicategories and fix $y \in \mathcal{D}$. An object $x \in \mathcal{C}$ is a right adjoint object to y under L if there exists an equivalence $\mathrm{Mor}_{\mathcal{C}}(-, x) \simeq \mathrm{Mor}_{\mathcal{D}}(L(-), y)$ in $\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}}, \mathsf{Ani})$.

Definition 3.2 (Right Adjoint; [Wag24, Def. 6.1])**.** Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of quasicategories. A functor $R : \mathcal{D} \to \mathcal{C}$ is a right adjoint of L if there exists an equivalence $\mathrm{Mor}_{\mathcal{C}}(-, R(-)) \simeq \mathrm{Mor}_{\mathcal{D}}(L(-), -)$ in $\mathrm{Fun}(\mathcal{C}^{\mathsf{Opp}} \times \mathcal{D}, \mathsf{Ani})$.

As usual, we will write $L \dashv R$.

It turns out, we can assemble a right adjoint from right adjoint objects.

Lemma 3.3 (Pointwise Right Adjoints; [Wag24, Lem. 6.2])**.** Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of quasicategories. L admits a right adjoint R if and only if each $y \in \mathcal{D}$ admits a right adjoint object $x \in \mathcal{C}$.

We have already seen examples of adjoints.

Example 3.4 (Adjoints to $\mathsf{Ani} \hookrightarrow \mathsf{Cat}_\infty$; [Wag24, Ex. 6.3])**.** The inclusion of animae to ∞-categories admits both a left and right adjoint. The right adjoint is given by the restriction to the core functor $\mathrm{core} : \mathsf{Cat}_\infty \to \mathsf{Ani}$ and the left adjoint is given by localization at all non-isomorphisms.

Remark 3.5. The adjunction behavior of Example 3.4 should not seem out of hand.

- (That core is a right adjoint) Forgetful functors are typically right adjoints.
- (That localization is the left adjoint) Localization/group completion has a universal property of "mapping from." Compare, for example, the universal property of group completion or ring localization in the 1-categorical case.

This insight is deeply indebted to Mike Hopkins who once remarked in class something along the lines of "there are two types of mathematical objects – ones good for mapping into and ones good for mapping out of."

As in the 1-categorical case, we can produce units and counits which turn out to be necessary and sufficient for the existence of adjoints as in the case of 1-categories.

Proposition 3.6 ([Wag24, Cons. 6.4, Lem. 6.5])**.** Let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$ be functors between quasicategories. L is a left adjoint to R if and only if there are natural transformations $u : \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L, c : L \circ R \Rightarrow \mathrm{id}_{\mathcal{D}}$ and equivalences $i_L : L \to L, i_R : R \to R$ making the diagrams

$$\begin{array}{ccc} L & \xrightarrow{Lu} & L \circ R \circ L \\ & \searrow^{i_L} & \downarrow{\scriptstyle cL} \\ & & L \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{uR} & R \circ L \circ R \\ & \searrow^{i_R} & \downarrow{\scriptstyle Rc} \\ & & R \end{array}$$

commute.

When working with ∞-categories it is often useful to restrict our attention to functors between ∞-categories which are left (resp. right) adjoints. This leads to the following definition.

Definition 3.7 ($\mathsf{Cat}_\infty^{\mathsf{L}}, \mathsf{Cat}_\infty^{\mathsf{R}}$; [Wag24, Corr. 6.8])**.** The ∞-category $\mathsf{Cat}_\infty^{\mathsf{L}}$ (resp. $\mathsf{Cat}_\infty^{\mathsf{R}}$) is the ∞-category by zero simplices ∞-categories and mapping animae the core $\mathrm{coreFun}^{\mathsf{L}}(\mathcal{C}, \mathcal{D})$ of those functors $\mathcal{C} \to \mathcal{D}$ which are left adjoints (resp. $\mathrm{coreFun}^{\mathsf{R}}(\mathcal{C}, \mathcal{D})$ which are right adjoints).

This yields the following – perhaps abtruse – result that will play an important role in defining Pr^{L} which plays a key role in ∞-categorical algebra.

Proposition 3.8 ([Wag24, Corr. 6.8 (b)])**.** There is an equivalence of ∞-categories betewen $\mathsf{Cat}_\infty^{\mathsf{L}}$ and $(\mathsf{Cat}_\infty^{\mathsf{R}})^{\mathsf{Opp}}$ which it is the identity on objects and taking left adjoints to their right adjoints.

4. (Co)Limits in ∞-Categories

This section follows [Wag24, §6.1] fairly closely.

Definition 4.1 (Colimit; [Wag24, Def. 6.9])**.** Let $F : \mathcal{I} \to \mathcal{C}$ be a functor between quasicategories. A colimit $\operatorname{colim} F$ of F is a left adjoint object of F under the constant functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.

Definition 4.2 (Limit; [Wag24, Def. 6.9])**.** et $F : \mathcal{I} \to \mathcal{C}$ be a functor between quasicategories. A colimit $\lim F$ of F is a right adjoint object of F under the constant functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.

Let us unpack why this is reasonable. Being a left adjoint object, dual to Definition 3.1, implies that there is an equivalence of animae

$$\mathrm{Mor}_{\mathcal{C}}(\operatorname{colim} F, y) \simeq \mathrm{Mor}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, \mathrm{const}(y)). \tag{4.1}$$

Loosely, we can think of morphisms $F \to \mathrm{const}(y)$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ as maps from $F(i) \to y$ for each $i \in \mathcal{I}$ compatible with the morphisms in $\mathcal{I}$ – indeed, this is exactly what happens in 1-categories (vis. eg. [Stacks, Tag 001I]). The equivalence of the left and right hand side of (4.1) captures the desired universal property:

- Any compatible system of maps from the $\mathcal{I}$-indexed diagram F in $\mathcal{C}$ to y factors uniquely over the colimit $\operatorname{colim} F$.

One readily makes an analogous argument for limits under passage to opposites, realizing them as the "universal object over diagrams" in an appropriate sense. We will provide a description of limits and colimits solidifiying this intuition in Section 5.

Adjoint functors behave with limits and colimits in an expected manner.

Lemma 4.3 (LAPC/RAPL; [Wag24, Lem. 6.11])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If F is a left adjoint (resp. is a right adjoint) then F preserves colimits (resp. preserves limits).

Moreover, it turns out that limits and colimits in Cat_∞ can be computed in terms of straightenings and unstraightenings. For this, we use the following results as a black box.

This will be discussed in the talk on presentable and accessible ∞-categories in a few weeks.

In particular, computations in Pr^{L} are often done "by passage to right adjoints."

Ie. taking $x \in \mathcal{C}$ to the functor that returns x on each $i \in \mathcal{I}$. This is a "left" thing because colimits are good for "mapping from" as in the philosophical sidebar of Remark 3.5 above.

Lemma 4.4 (Pointwise Limits; [Wag24, Lem. 6.12])**.** Let $\mathcal{C}, \mathcal{D}, \mathcal{I}$ be quasicategories such that $\mathcal{D}$ has all $\mathcal{I}$-indexed limits. Then the functor quasicategory $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ has all $\mathcal{I}$-indexed limits and

$$\mathrm{ev}_x : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\{x\}, \mathcal{D})$$

preserves $\mathcal{I}$-indexed colimits.

Lemma 4.5 ([Wag24, Lem. 6.15])**.** Let $\mathcal{C}$ be a quasicategory. The functor $\mathsf{Cat}_\infty \to \mathsf{Ani}$ by $\mathcal{D} \mapsto \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{C}, \mathcal{D}) = \mathrm{coreFun}(\mathcal{C}, \mathcal{D})$ preserves pullbacks.

Limits and colimits in Cat_∞ are given by straightenings and unstraightenings as follows.

Proposition 4.6 ([Wag24, Lem. 6.14])**.** Let $F : \mathcal{I} \to \mathsf{Cat}_\infty$ be a functor with coCartesian unstraightening $p : \mathcal{E} \to \mathcal{I}$. Then

$$\operatorname{colim} F \simeq \mathcal{E}[\{\text{coCartesian edges}\}^{-1}] \text{ and } \lim F = \mathrm{Fun}_{\mathcal{I}}^{\mathsf{coCart}}(\mathcal{I}, \mathcal{E}).$$

Ie. the limit is given by coCartesian sections of p, resp. the localization at all non-isomorphisms.

Furthermore, if F is Ani-valued, then the limit and colimit are given by

$$\operatorname{colim} F \simeq |\mathcal{E}| \text{ and } \lim F = \mathrm{Mor}_{\mathsf{Cat}_{\infty/\mathcal{I}}}(\mathcal{I}, \mathcal{E}).$$

Proof Outline. Considering the case of colimits, let F be Ani-valued. Under the straightening-unstraightening correspondence, we have

$$\mathrm{Mor}_{\mathrm{Fun}(\mathcal{I},\mathsf{Ani})}(F, \mathrm{const}(-)) \simeq \mathrm{Mor}_{\mathsf{LFib}(\mathcal{I})}(\mathcal{E}, \mathrm{Un}(\mathrm{const}(-)))$$

but noting that the unstraightening of a constant functor $\mathrm{const}(x)$ is the projection $\{x\} \times \mathcal{I} \to \mathcal{I}$. As such we can compute

$$\begin{aligned}
\mathrm{Mor}_{\mathsf{LFib}(\mathcal{I})}(\mathcal{E}, - \times \mathcal{I}) &\simeq \mathrm{Mor}_{\mathsf{Cat}_{\infty/\mathcal{I}}}(\mathcal{E}, - \times \mathcal{I}) && \text{(i)}\\
&\simeq \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E}, - \times \mathcal{I}) \times_{\mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E},\mathcal{I})} \{p\} && \text{(ii)}\\
&\simeq (\mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E}, -) \times \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E}, \mathcal{I})) \times_{\mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E},\mathcal{I})} \{p\} && \text{(iii)}\\
&\simeq \mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E}, -) && \text{(iv)}
\end{aligned}$$

where

(i) The inclusion $\mathsf{LFib}(\mathcal{I}) \to \mathsf{Cat}_{\infty/\mathcal{I}}$ is fully faithful. This is reasonable because being a left fibration is a property. (Cf. the sidebar of (2.3)).
(ii) Pointwise computation of colimits in functor categories Lemma 4.4 allow us to reduce this to a result on homotopy pullbacks shown last week [Wag24, Corr. 5.15].
(iii) This uses the product-map adjunction in combination with the fact that forgetting to animae preserves pullbacks Lemma 4.5.
(iv) The fibered product $\mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E}, \mathcal{I}) \times_{\mathrm{Mor}_{\mathsf{Cat}_\infty}(\mathcal{E},\mathcal{I})} \{p\}$ is just $\{p\}$.

Now localization at non-isomorphisms is left adjoint to the inclusion as in Example 3.4 and $|\mathcal{E}|$ satisfies the desired universal property, giving the claim. ■

5. Excursus: Some Handwaving with Slices

This is indeed the way the author first learned about (co)limits in ∞-categories.

This section largely follows Hopkins' course (and associated non-public notes) [Hop23] on which the author's course notes [—23] are based. This is also how things were originally defind by Joyal. Some of these constructions will also be revisited in the proof of the adjoint functor theorem.

Let us recall the slice construction of [Wag24, §2.11]. The slice under quasicategory is given by the Cartesian square

$$\begin{array}{ccc} \mathcal{C}_{x/} & \longrightarrow & \operatorname{Arr}(\mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \{x\} \times \mathcal{C} & \longrightarrow & \mathcal{C} \times \mathcal{C}. \end{array}$$

I am playing a bit fast and loose here. See the discussion in Wagner's notes, but everything is justified via the equivalences between slices and "fat slices."

Dually, the slice over quasicategory is given by the Cartesian square

$$\begin{array}{ccc} \mathcal{C}_{/x} & \longrightarrow & \operatorname{Arr}(\mathcal{C}) \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{C} \times \{x\} & \longrightarrow & \mathcal{C} \times \mathcal{C}. \end{array}$$

Slice constructions allow us to define initial and terminal objects.

Definition 5.1 (Initial/Terminal Objects)**.** Let $\mathcal{C}$ be a quasicategory. An object $x \in \mathcal{C}$ is initial (resp. terminal) if $\mathcal{C}_{x/} \to \mathcal{C}$ (resp. $\mathcal{C}_{/x} \to \mathcal{C}$) is a trivial fibration.

$\mathcal{C}_{x/} \to \mathcal{C}, \mathcal{C}_{/x} \to \mathcal{C}$ are left and right fibrations, resp.

This should be believable because commutativity of the diagram gives structure maps $x \to y$ in $\mathcal{C}_{x/}$ or $y \to x$ in $\mathcal{C}_{/x}$ for fixed x.

Lemma 5.2. Let $\mathcal{C}$ be a quasicategory. The full subcategory spanned by initial or final objects are either empty or contractible animae.

Also recall that trivial fibrations are the right notions of uniqueness in ∞-land.

Now let's suppose we can construct slices not just over individual objects of a quasicategory but over the image of a functor $\mathcal{I} \to \mathcal{C}$ – ie. that exhibit an $\mathcal{I}$-indexed diagram as initial or final presented as a system of maps to/from objects of the diagram, compatible with morphisms in the diagram – and moreover that this is in fact a quasicategory. This can be done by writing down explicit simplicial sets.

To wit, for $F : \mathcal{I} \to \mathcal{C}$ a functor of quasicategories,

- $\mathcal{C}_{\mathcal{I}/}$ is the quasicategory of cocones parametrizing objects under the $\mathcal{I}$-indexed diagram.
- $\mathcal{C}_{/\mathcal{I}}$ is the quasicategory of cones parametrizing objects over the $\mathcal{I}$-indexed diagram.

This allows us to very easily define limits and colimits.

Definition 5.3 ((Co)limit)**.** Let $F : \mathcal{I} \to \mathcal{C}$ be a functor of quasicategories. A colimit of F is an initial object of the slice quasicategory $\mathcal{C}_{\mathcal{I}/}$ (resp. a limit of F is a final object of the slice quasicategory $\mathcal{C}_{/\mathcal{I}}$).

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